

# A NOTE ON EULER NUMBER AND POLYNOMIALS

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We investigate some properties of non-Archimedean integration which is defined by Kim. By using our results in this paper, we can give an answer to the problem which is introduced by I.-C. Huang and S.-Y. Huang in 1999.

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## 1. Introduction

Throughout this paper  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ , and  $\mathbb{C}_p$  will, respectively, denote the ring of  $p$ -adic rational integers, the field of  $p$ -adic rational numbers, and the completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $v_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-v_p(p)} = p^{-1}$ .

Let  $p$  be a fixed prime number and let  $l$  be a fixed integer with  $(p, l) = 1$ . We set

$$\begin{aligned} X &= \varprojlim_N (\mathbb{Z}/lp^N\mathbb{Z}), \\ X^* &= \bigcup_{\substack{0 < a < lp \\ (a, p) = 1}} (a + lp\mathbb{Z}_p), \\ a + lp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{lp^N}\}, \end{aligned} \tag{1.1}$$

where  $a \in \mathbb{Z}$  lies in  $0 \leq a < lp^N$  (cf. [3, 4]).

For any positive integer  $N$ , we set

$$\mu_1(a + lp^N\mathbb{Z}_p) = \frac{1}{lp^N} \tag{1.2}$$

and this can be extended to a distribution on  $X$  (see [3, 9]).

## 2 A note on Euler number and polynomials

This distribution yields an integral for nonnegative integer  $m$ :

$$\int_X x^m d\mu_1(x) = B_m, \quad (1.3)$$

where  $B_m$  are called usual Bernoulli numbers (cf. [8]).

The Euler numbers  $E_m$  are defined by the generating function in the complex number field as follows:

$$\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!} \quad (|t| < \pi) \quad (1.4)$$

where we use the technique method notation by replacing  $E^m$  by  $E_m$  ( $m \geq 0$ ), symbolically (cf. [3, 5, 7, 9, 10]).

The Bernoulli numbers with order  $k$ ,  $B_n^{(k)}$ , were defined by

$$\left( \frac{t}{e^t - 1} \right)^k = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \quad (\text{cf. [5, 10]}). \quad (1.5)$$

Let  $u$  be algebraic in complex number field. Then Frobenius-Euler numbers were defined by

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!} \quad (\text{cf. [5]}). \quad (1.6)$$

By (1.4) and (1.6), note that  $H_n(-1) = E_n$ .

In this paper, we will give the interesting formulae for sums of products of Euler numbers (= Frobenius-Euler numbers) by using  $p$ -adic Euler integration which is defined in [3, 5, 8–10]. Our result is an answer to the problem which is introduced by I.-C. Huang and S.-Y. Huang in [2, page 179].

### 2. Sums of products of Euler numbers

Let  $u \in \mathbb{C}_p$  with  $|1 - u^f|_p \geq 1$  for each positive integer  $f$ . Then the  $p$ -adic Euler measure was defined by

$$E_u(x) = E_u(x + d p^N \mathbb{Z}_p) = \frac{u^{d p^N - x}}{1 - u^{d p^N}}, \quad (\text{cf. [3, 5]}). \quad (2.1)$$

Now, we define Euler polynomials with order  $n$  by

$$\left( \frac{u}{1-u} \right)^m H_n^{(m)}(u, x) = \underbrace{\int_X \cdots \int_X}_{m \text{ times}} (x + x_1 + \cdots + x_m)^n dE_u(x_1) \cdots dE_u(x_m). \quad (2.2)$$

In the case  $x = 0$ , we use the following notations:

$$H_n^{(k)}(u, 0) = H_n^{(k)}(u), \quad H_n^{(1)}(u) = H_n(u) \quad (\text{cf. [3, 9]}). \quad (2.3)$$

In [3], the following formula can be found:

$$\int_{\mathbb{Z}_p} x^n dE_u(x) = \frac{u}{1-u} H_n(u). \quad (2.4)$$

By (2.2) and (2.4), we easily see that  $\lim_{k \rightarrow 1} H_n^{(k)}(u) = H_n(u)$ .

For any positive integer  $m$ ,  $H_n^{(m)}(u, x)$  can be written by

$$H_n^{(m)}(u, x) = \sum_{j=0}^n \binom{n}{j} x^{n-j} H_j^{(m)}(u). \quad (2.5)$$

We may now mention the following formulae which are easy to prove:

$$\left( \frac{u}{1-u} \right)^m H_n^{(m)}(u, x) = l^n \sum_{l_1, \dots, l_m=0}^{l-1} \frac{u^{ml - \sum_{i=1}^m l_i}}{(1-u^l)^m} H_n^{(m)}\left(u^l, \frac{x + l_1 + \dots + l_m}{l}\right), \quad (2.6)$$

where

$$\sum_{l_1, \dots, l_m=0}^{l-1} = \sum_{l_1=0}^{l-1} \sum_{l_2=0}^{l-1} \dots \sum_{l_m=0}^{l-1}. \quad (2.7)$$

By using (2.2) and multinomial coefficients, We obtain the following theorem.

**THEOREM 2.1.** For  $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{C}_p$  and positive integers  $n, m$ ,

$$H_n^{(m)}(u, \alpha_1 + \alpha_2 + \dots + \alpha_m) = \sum_{\substack{i_1, \dots, i_m \\ n=i_1+\dots+i_m}} \binom{n}{i_1, \dots, i_m} H_{i_1}(u, \alpha_1) H_{i_2}(u, \alpha_2) \dots H_{i_m}(u, \alpha_m), \quad (2.8)$$

where  $\binom{n}{i_1, \dots, i_m}$  is the multinomial coefficient.

**Remark 2.2.** The above theorem is an answer to the problem which was introduced in [2, page 179].

**Remark 2.3.** Note that  $H_n(-1) = \sum_{k=0}^n \binom{n+1}{k} 2^k B_k$ , where  $B_k$  are the  $k$ th ordinary Bernoulli numbers.

**Remark 2.4.** By using Volkenborn integral, it was well known that

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_1(x) \frac{t^n}{n!} \quad (\text{cf. [3, 7, 10]}). \quad (2.9)$$

In [1, 9], note that

$$\left(\frac{t}{e^t - 1}\right)^k = \sum_{n=0}^{\infty} \underbrace{\iint_X \cdots \int_X}_{k \text{ times}} (x + x_1 + \cdots + x_k)^n d\mu_1(x_1) d\mu_1(x_2) \cdots d\mu_1(x_k) \frac{t^n}{n!}. \quad (2.10)$$

The Bernoulli polynomials with order  $k$ ,  $B_n^{(k)}(x)$ , were defined by

$$B_n^{(k)}(x) = \underbrace{\iint_X \cdots \int_X}_{k \text{ times}} (x + x_1 + \cdots + x_k)^n d\mu_1(x_1) d\mu_1(x_2) \cdots d\mu_1(x_k) \quad (\text{cf. [7, 9, 10]}). \quad (2.11)$$

In the case  $x = 0$ , we write  $B_n^{(k)}(0) = B_n^{(k)}$  (cf. [9]).

In [2], the authors proved the formulae of sums of products of Bernoulli numbers of higher order by using theory of residues. By using the properties of invariant  $p$ -adic integrals in this paper, we can also give the same formulae on the sums of products for  $B_n^{(k)}$  in [2]. Let  $\chi$  be a Dirichlet character with conductor  $f$ . We set  $p^* = p$  for  $p \geq 2$ , and  $p^* = 4$  for  $p = 2$ . Let  $\tilde{f} = (f, p^*)$  be denoted by the least common multiple of the conductor  $f$  of  $\chi$  and  $p^*$ .

Now, we define the generalized Bernoulli numbers of higher order with  $\chi$  as

$$B_{n,\chi}^{(m)} = \int_X \cdots \int_X \chi(x_1 + \cdots + x_m) (x_1 + \cdots + x_m)^n d\mu_1(x_1) \cdots d\mu_1(x_m). \quad (2.12)$$

We easily get in (2.12)

$$B_{n,\chi}^{(m)} = l^{n-m} \sum_{x_1, \dots, x_m=0}^{l-1} B_n^{(m)}\left(\frac{x_1 + \cdots + x_m}{l}\right) \chi(x_1 + \cdots + x_m), \quad (2.13)$$

where  $B_{n,\chi}$  is the generalized ordinary Bernoulli number with  $\chi$ .

By (2.12), we have

$$B_{n,\chi}^{(m)} = \lim_{\rho \rightarrow \infty} \frac{1}{(\tilde{f} p^\rho)^m} \sum_{1 \leq x_1 \leq \tilde{f} p^\rho} \cdots \sum_{1 \leq x_m \leq \tilde{f} p^\rho} \chi(x_1 + \cdots + x_m) (x_1 + \cdots + x_m)^n. \quad (2.14)$$

The investigation of these numbers is left to the interested reader.

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